MMAT5520 Differential Equations & Linear Algebra
Final Exam
2 Dec 2014
Time allowed: 120 mins

Answer all questions.

1. (10 marks) Let $L[y] = y'' - 2y' - 3y$.
   (a) Find the general solution of the homogeneous equation $L[y] = 0$
   (b) Use the method of variation of parameter to find a particular solution to the non-homogeneous equation $L[y] = 4e^{3t}$.

2. (10 marks) Let $L[y] = y''' - 9y'$.
   (a) Find a fundamental set of solutions to the homogeneous equation $L[y] = 0$ and write down the Wronskian of the set of solutions.
   (b) Write down an appropriate form of a particular solution (do not solve the equation) to the equation

$$L[y] = 3t^2 - 2te^{-3t} + 5\cos t$$

3. (10 marks) Find the general solution of the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}$$

4. (15 marks) Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices.
   (a) Suppose $\mathbf{A}$ is invertible. Prove that if $\mathbf{A}$ is diagonalizable, then $\mathbf{A}^{-1}$ is also diagonalizable.
   (b) Suppose $\mathbf{A}$ is invertible. Prove that $\mathbf{AB}$ and $\mathbf{BA}$ have the same minimal polynomial.
   (c) Find two 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB}$ and $\mathbf{BA}$ have different minimal polynomials.

5. (15 marks) Let

$$\mathbf{A} = \begin{pmatrix} -4 & 0 & -3 \\ 6 & 2 & 3 \\ 6 & 0 & 5 \end{pmatrix}$$

   (a) Diagonalize $\mathbf{A}$.
   (b) Find the minimal polynomial of $\mathbf{A}$.
   (c) Express $\mathbf{A}^4$ and $\mathbf{A}^{-1}$ as a polynomial of $\mathbf{A}$ of smallest degree.
   (d) Find $\exp(\mathbf{A}t)$.

6. (15 marks) Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & 5 \end{pmatrix}$ and $\mathbf{J}$ be the Jordan normal form of $\mathbf{A}$.

   (a) Find a generalized eigenvector of rank 3 of $\mathbf{A}$.

   (b) Find the general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

   (c) Find $\mathbf{J}$ and an invertible matrix $\mathbf{Q}$ such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{J}$.

7. (15 marks) Let

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$$

   (a) Find the Jordan norm form of $\mathbf{A}$.

   (b) Find $\exp(\mathbf{A}t)$.

   (c) Find a fundamental matrix $\Psi(t)$ for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\Psi(0) = \begin{pmatrix} 3 & 0 \\ 1 & -4 \end{pmatrix}$

   (d) Find $\mathbf{A}^{99}$.

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